The gauge field copies and Backlund transformation



1987 J. Phys. A: Math. Gen. 20 2685

(http://iopscience.iop.org/0305-4470/20/10/012)

J. Phys. A: Math. Gen. **20** (1987) 2685-2691. Printed in the UK

# The gauge field copies and Bäcklund transformation

Ziemowit Popowicz†
Dipartimento di Fisica, Università di Roma 'La Sapienza', Roma, Italy

Received 30 June 1986

**Abstract.** We show explicitly that the famous auto Bäcklund transformation and non-auto Kruskal-Dodd-Bullough transformation for the sine-Gordon equation can be interpreted as the gauge field copies for the strength tensor associated with the Lax pair for the sine-Gordon equation. In the same way we interpret the auto and non-auto Bäcklund transformation for the Liouville equation.

## 1. Introduction

Bäcklund introduced his transformation [1] as the transformation mapping one pseudospherical surface into another. Originally Bäcklund considered the line element of a surface of constant negative curvature which can be written as

$$(\mathrm{d}s)^2 = \alpha^2[(\mathrm{d}x)^2 + 2\,\mathrm{d}x\,\mathrm{d}y\cos\omega + (\mathrm{d}y)^2] \tag{1.1}$$

where $-1/\alpha^2$ is the constant total curvature of the surface and $\omega$ is the angle between the asymptotic lines satisfying the celebrated sine-Gordon equation

$$\frac{\partial^2}{\partial x \partial y}\omega = \sin\omega. \tag{1.2}$$

It was found by Bäcklund that a new solution (i.e. surface) $\omega_1$ could be obtained from a given solution by means of the relation

$$\frac{\partial}{\partial x}(\omega_1 - \omega) = 2\lambda \sin\left(\frac{\omega_1 + \omega}{2}\right) \tag{1.3}$$

$$\frac{\partial}{\partial y}(\omega_1 + \omega) = \frac{2}{\lambda}\sin\left(\frac{\omega_1 - \omega}{2}\right) \tag{1.4}$$

where $\lambda$ is an arbitrary constant.

The sine-Gordon equation belongs to the special class of non-linear partial differential equations which have attracted the attention of physicists for a long time. These equations, the so-called soliton equations, have important physical applications and share several remarkable properties.

† Permanent address: Institute of Theoretical Physics, University of Wrocław, ul Cybulskiego 36, 50-205 Wrocław, Poland.

This work has been done in the framework of the exchange programme between Wrocław and Rome University 'La Sapienza'.

  

(i) The initial value problem can be solved exactly in terms of linear procedures by the use of the so-called 'inverse scattering transformation'.

(ii) They have infinite numbers of 'conservation laws'.

(iii) They have a 'Bäcklund transformation'.

(iv) They describe pseudospherical surfaces.

Many papers in the last decade considered how these properties are connected amongst themselves [1-13]. Here we would like to consider how the Bäcklund transformation is connected with the inverse scattering transformation. This method was first devised for the Korteweg-de Vries equation [2, 3]. Later it was extended by Zakharov and Shabat to a $2\times 2$ scattering problem for the non-linear Schrödinger equation [4] and was subsequently generalised by Ablowitz *et al* (AKNS) [5] to include a variety of non-linear equations. The heart of the AKNS framework is the so-called Lax pair

$$\psi_+ = \frac{\partial}{\partial x_+}\psi = A_+\psi \tag{1.5}$$

$$\psi_- = \frac{\partial}{\partial x_-}\psi = A_-\psi \tag{1.6}$$

where $\psi$ is a two-column vector and $A_\pm = A_\pm(f, f_+, f_-, f_{+-}, \ldots, \lambda)$ are $2\times 2$ matrices depending on some function $f = f(x_+, x_-)$, its derivatives and on the spectral parameter $\lambda$ whose integrability condition

$$\partial_- A_+ - \partial_+ A_- + [A_+, A_-] = 0 \tag{1.7}$$

gives us the broad class of the two-dimensional non-linear partial differential equations on the function $f$.

Chen [6] and Wadati *et al* [7] showed that many Bäcklund transformations for the soliton equations can be derived from the set of Riccati equations obtained from the inverse scattering transformation. The connection between those two methods can also be formulated in the geometrical language.

The geometrical interpretation of the inverse scattering transformation can be achieved [8, 9] by association of a pair of completely integrable Pfaffian equations

$$\mathrm{d}v = \Omega v \tag{1.8}$$

where $v$ is a two-column vector and d denotes the exterior derivative. The $2\times 2$ matrix $\Omega$ is a one-parameter ($\lambda$ is the eigenvalue) family of 1-forms in the independent variables $x_+$, $x_-$. Integrability of (1.8)

$$0 = \mathrm{d}^2 v = \mathrm{d}\Omega v - \Omega \wedge \mathrm{d}v = (\mathrm{d}\Omega - \Omega \wedge \Omega)v \tag{1.9}$$

requires the vanishing of the 2-form

$$\theta = \mathrm{d}\Omega - \Omega \wedge \Omega \tag{1.10}$$

which plays the same role as (1.7). Sasaki [10] has shown that in this geometrical framework the Bäcklund transformation which maps one pseudospherical surface into another is obtained from $\Omega$ by the gauge transformation

$$v' = Av \qquad \Omega' = \mathrm{d}AA^{-1} + A\Omega A^{-1} \tag{1.11}$$

$$\theta' = A\theta A^{-1} \tag{1.12}$$

where $A$ is an arbitrary matrix with the determinant unity.

On the other hand, Levi *et al* [11, 12] has shown that the Bäcklund transformation can be interpreted as the gauge transformation of the Lax pair—analogous to the geometrical approach.

These methods apply to all known auto Bäcklund transformations (e.g. the transformation mapping solutions of a given equation into solutions of the same equation) for the equations obtained from the AKNS scheme. The present author has shown [13] that the non-auto Bäcklund transformation (e.g. the transformation mapping solutions of a given equation into solutions of the different equation) for the Liouville equation can be interpreted as the transformation which relates two different gauge copies of the field strength tensor associated with the Lax pair for the Liouville equation. In this paper we would like to continue this approach and we explicitly show also that the auto Bäcklund transformation for the Liouville and sine-Gordon equations (see (1.3) and (1.4)) can be interpreted in a similar manner. Also we show that the non-auto Kruskal–Dodd–Bullough transformation [14] for the sine-Gordon equation has the same structure.

The gauge field copies appear in the gauge theories and describe the surprising phenomenon that two non-singular potentials can give rise to identical field strengths ('copies') without being gauge equivalent. Since its discovery [15] and the discussion of necessary conditions for its occurrence [16] a number of further examples have been given [17–19] and there is now a sizable literature on this subject [19–21]. In [21–25] several authors discussed the construction of field copies and the criteria for determining which connections are defined uniquely by their curvature.

Let us apply the concept of the gauge field copies to the Lax pair (1.5) and (1.6). First let us notice that, besides considering the integrability condition (1.7), it is possible to introduce the two-dimensional field strength tensor for the Lax pair by

$$F_{+-}=\partial_+A_- - \partial_-A_+ + [A_+, A_-] \tag{1.13}$$

where $A_\pm$ are the same matrix functions as in (1.5) and (1.6). It is well known that under the gauge transformation of the potentials

$$A'_\pm = g^{-1}A_\pm g + g^{-1}\partial_\pm g \tag{1.14}$$

where $g$ belongs to some gauge group, the field strength tensor transforms as

$$F'_{+-} = g^{-1}F_{+-}g. \tag{1.15}$$

The main problem in the gauge field copies is to find two non-gauge equivalent potentials which give the same field strength tensor. In the general case let us assume that we have two potentials $A_\pm$ and $A'_\pm$ where

$$A'_\pm = A_\pm + A^0_\pm. \tag{1.16}$$

Then the condition that these potentials produce the same field strength tensor is

$$F'_{+-} = F_{+-} = \partial_+A'_- - \partial_-A'_+ + [A'_+, A'_-] \tag{1.17}$$

and reduces to the following system of equations for the matrix functions:

$$\partial_+A^0_- - \partial_-A^0_+ + [A_+, A^0_-] + [A^0_+, A_-] + [A^0_+, A^0_-] = 0. \tag{1.18}$$

In the next section we consider the equations (1.16) and (1.18) for the Liouville and sine-Gordon equations in order to find the matrices $A^0_\pm$ which give us the same strength tensor and we show that they are generated by the different Bäcklund transformations. In the appendix we prove that these potentials are gauge non-equivalent.

## 2. The gauge field copies for the Liouville and sine-Gordon equations

For the Liouville equation we have the following AKNS representation:

$$\psi_+ = (\lambda\sigma_3 + f_+\sigma_1)\psi = A_+\psi \tag{2.1}$$

$$\psi_- = (e^{2f}/2\lambda)(\sigma_3 - i\sigma_2)\psi = A_-\psi \tag{2.2}$$

where $\sigma_1$, $\sigma_2$, $\sigma_3$ are the Pauli matrices and $\lambda$ is the spectral parameter. The field strength tensor associated with this equation and this Lax pair is

$$F_{+-} = \frac{2f_+}{\lambda} e^{2f}(\sigma_3 - i\sigma_2) - (e^{2f} + f_{+-})\sigma_1. \tag{2.3}$$

Assuming the following form of $A^0_\pm$

$$A^0_+ = a\sigma_3 + i\sigma_2 b \tag{2.4}$$

$$A^0_- = K\sigma_1 \tag{2.5}$$

where $a$, $b$, $K$ are some functions, then equation (1.18) reduces to

$$\partial_- a = 2bK \tag{2.6}$$

$$\partial_- b = 2\lambda K + 2aK \tag{2.7}$$

$$\partial_+ K = (a+b)e^{2f}/\lambda. \tag{2.8}$$

One can easily prove that the following functions

(a)

$$a = \lambda \sinh 2(g-f) - \lambda \tag{2.9}$$

$$b = \lambda \cosh 2(g-f) \tag{2.10}$$

$$K = \lambda \exp(f+g) \tag{2.11}$$

satisfy (2.6)-(2.8) and are equivalent with the auto Bäcklund transformation

$$\partial_-(g-f) = \lambda \exp(f+g) \tag{2.12}$$

$$\partial + (g+f) = (2/\lambda)\sinh(g-f) \tag{2.13}$$

for which the integrability condition give us the Liouville equation for $f$ as well as for $g$.

(b)

$$a = \exp[2(f+v)] - \lambda \tag{2.14}$$

$$b = -\exp[2(f+v)] \tag{2.15}$$

$$K = -\lambda \exp(f-v) \tag{2.16}$$

satisfy (2.6)-(2.8) too and are equivalent with the non-auto Bäcklund transformation

$$\partial_+(f-v) = (1/\lambda)\exp(f+v) \tag{2.17}$$

$$\partial_-(f+v) = \lambda \exp(f-v) \tag{2.18}$$

for which the integrability condition gives us the Liouville equation for $f$ and the two-dimensional Laplace equation for $v$.

For the sine-Gordon equation we have the following AKNS representation

$$\psi_+ = (\eta\sigma_3 - \tfrac{1}{2}\mathrm{i}\omega_+\sigma_2)\psi = A_+\psi \tag{2.19}$$

$$\psi_- = \left(\frac{\cos\omega}{4\eta}\sigma_3 + \frac{\sin\omega}{4\eta}\sigma_1\right)\psi = A_-\psi \tag{2.20}$$

$$F_{+-} = \frac{\omega_+}{2\eta}\cos\omega\sigma_1 - \tfrac{1}{2}\mathrm{i}\,(\omega_{+-} + \sin\omega)\sigma_2 - \frac{\omega_+}{2\eta}\sin\omega\sigma_3 \tag{2.21}$$

where now $\eta$ is the spectral parameter.

Assuming the following form of $A^0_\pm$

$$A^0_+ = a\sigma_3 + b\sigma_1 \tag{2.22}$$

$$A^0_- = K\sigma_2 \tag{2.23}$$

the system (1.18) reduces to

$$\partial_- a = 2\mathrm{i}Kb \tag{2.24}$$

$$-\partial_- b = 2\mathrm{i}\eta K + 2\mathrm{i}aK \tag{2.25}$$

$$2\eta\partial_+ K = \mathrm{i}(b\cos\omega - a\sin\omega). \tag{2.26}$$

Now one can easily check that

(a)

$$a = \eta\cos(\omega_1 + \omega) - \eta \tag{2.27}$$

$$b = \eta\sin(\omega_1 + \omega) \tag{2.28}$$

$$K = \mathrm{i}\lambda\sin\tfrac{1}{2}(\omega_1 - \omega) \tag{2.29}$$

satisfy (2.24)-(2.26) and are equivalent with the auto Bäcklund transformation for the sine-Gordon equation (1.2)-(1.4).

(b)

$$a = \eta\cos 2\omega_1 - \eta \tag{2.30}$$

$$b = \eta\sin 2\omega_1 \tag{2.31}$$

$$K = (\mathrm{i}/\lambda)\sin(\omega_1 - \omega) \tag{2.32}$$

satisfy (2.24)-(2.26) too and are equivalent with the non-auto Kruskal-Dodd-Bullough [14] transformation for the sine-Gordon equation

$$\partial_-\omega_1 = (1/\lambda)\sin(\omega_1 - \omega) \tag{2.33}$$

$$\partial_+(\omega_1 - \omega) = \lambda\sin\omega_1. \tag{2.34}$$

The integrability condition for (2.33) and (2.34) gives us the sine-Gordon equation for $\omega$ and $\omega_1$ satisfy

$$\partial_{+-}\omega_1 = [1 - (\lambda\partial_-\omega_1)^2]^{1/2}\sin\omega_1. \tag{2.35}$$

**Acknowledgments**

I would like to express my gratitude to Drs D Levi and O Ragnisco for stimulating discussions and their warm hospitality during my stay in Dipartimento di Fisica, Università 'La Sapienza'.

## Appendix. The gauge non-equivalence of the potentials $A_\pm$ and $A'_\pm$

It is a technical problem so we only outline the proof here. In order to prove that the potentials $A_\pm$ and $A'_\pm$ cannot be connected in our cases by the gauge transformation (1.14) we produce the contradiction. Let us assume that (1.14) and (1.15) is valid then we obtain the following lemma.

*Lemma.* Let $F_{+-}$ be such as in (1.13). Then in order to satisfy (1.15) and (1.17) $g$ should be

$$g = cF_{+-} + \chi 1 \tag{A1}$$

where $c$ and $\chi$ are arbitrary functions such that $\det g \neq 0$ and $g$ belongs to a some gauge group.

*Proof.* The condition (1.15) under assumption (1.17) is equivalent with $[g, F_{+-}] = 0$. Assuming that $g = a_i\sigma_i + \chi 1$, $i = 1, 2, 3$, and using $F_{+-} = F^i_{+-}\sigma_i$ which is valid for the AKNS system we easily recognise that (1.15) is equivalent to the homogeneous system of the algebraic equations on the functions $a_i$. The determinant of this system is equal to zero.

On the other hand, the condition (1.14) can be rewritten as

$$[g, A_\pm] + gA^0_\pm = \partial_\pm g. \tag{A2}$$

In order to find the contradiction in this formula we use (A1) for the particular cases considered in the previous section.

(a) Liouville equation. For the non-auto Bäcklund transformation from (A1) and (A2) it follows that

$$\chi = -c(e^{2f} + f_{+-}) \tag{A3}$$

$$\partial_+ \ln c(e^{2f} + f_{+-}) = -\frac{2f_+e^{2f}}{(e^{2f} + f_{+-})} \tag{A4}$$

$$\partial_- \ln c(e^{2f} + f_{+-}) = \lambda \exp(f - v) \tag{A5}$$

which is impossible because the integrability of (A4) and (A5) cannot be fulfilled.

For the auto Bäcklund transformation using (A1) and (A2) we obtain

$$\chi = c(e^{2f} + f_{+-}) \coth(g - f) \tag{A6}$$

which is in contradiction because also from (A2) it follows that for $f$ satisfying the Liouville equation we obtain

$$\partial_+(g - f) = -2f_+ e^{g-f} \sinh(g - f) \tag{A7}$$

which is not true.

(b) Sine-Gordon equation. From condition (A2) it follows that

$$\partial_-(cF^3_{+-}) = \mathrm{i}F^2_{+-}\, c \sin \omega/2\eta + \mathrm{i}cKF^1_{+-} \tag{A8}$$

$$\partial_-(cF^2_{+-}) = -\mathrm{i}cKF^3_{+-} - \mathrm{i}cF^2_{+-}\cos w/2\eta \tag{A9}$$

where $K$ and $F^i_{+-}$, $i = 1, 2, 3$, are defined by (2.29) or (2.32) and by (2.21) respectively. Equations (A8) and (A9) are in contradiction because from these formulae it follows that to satisfy the sine-Gordon equation we obtain

$$-\mathrm{i}K = \partial_-\omega \tag{A10}$$

which is not true both for auto and non-auto Bäcklund transformations.

## References


[1] Miura R M (ed) 1976 *Bäcklund Transformations.* (*Lecture Notes in Mathematics* **515**) (Berlin: Springer)
[2] Gardner C S, Green J M, Kruskal M D and Miura R M 1976 *Phys. Rev. Lett.* **19** 1095
[3] Scott A, Chu F and McLaughlin D 1973 *Proc. IEEE* **61** 1443
[4] Zakharov V E and Shabat A B 1972 *Sov. Phys.-JETP* **34** 62
[5] Ablowitz M, Kaup D, Newell A and Segur H 1973 *Phys. Rev. Lett.* **31** 125
[6] Chen Hsing-Hen 1974 *Phys. Rev. Lett.* **33** 925
[7] Wadati M, Sanuki H and Konno K 1975 *Prog. Theor. Phys.* **53** 419
[8] Dodd R K and Gibbon J D 1978 *Proc. R. Soc.* A **359** 411
[9] Crampin M 1978 *Phys. Lett.* **66A** 170
[10] Sasaki R 1979 *Nucl. Phys.* B **154** 343
[11] Levi D and Ragnisco O 1982 *Phys. Lett.* **87A** 381
[12] Levi D, Ragnisco O and Sym A 1982 *Lett. Nuovo Cimento* **33** 401
[13] Popowicz Z 1985 *Phys. Lett.* **111A** 336
[14] Bullough R K and Caudrey P J (ed) 1980 *Solitons* (Berlin: Springer)
[15] Wu T T and Yang C 1975 *Phys. Rev.* D **12** 3843
[16] Deser S and Wilczek F 1976 *Phys. Lett.* **65B** 391
[17] Roskies R 1977 *Phys. Rev.* D **15** 1731
[18] Calvo M 1977 *Phys. Rev.* D **15** 1733
[19] Halpern M 1979 *Phys. Rev.* D **19** 517
[20] Halpern M 1978 *Nucl. Phys.* B **139** 477
[21] Solomon S 1979 *Nucl. Phys.* B **147** 174
[22] Deser S and Drechsler W 1979 *Phys. Lett.* **86B** 189
[23] Bollini C, Giambiagi J and Tiomno J 1979 *Phys. Lett.* **83B** 185
[24] Doria F 1981 *Commun. Math. Phys.* **79** 435
[25] Mostow M and Shnider S 1983 *Commun. Math. Phys.* **90** 521